Quiz 2 Solutions

1. a) Consider the transformation $f(z) = 5z + \frac{4}{z}$ acting on the punctured complex plane $\mathbb{C} - \{0\}$. Find all fixed points of this transformation.

* A fixed point for a transformation f is a solution to the equation $f(z) = z$.

$$5z + \frac{4}{z} = z$$
$$4z + \frac{4}{z} = 0$$
$$4z^2 + 4 = 0$$
$$4(z^2+1) = 0$$
$$z^2 + 1 = 0$$

this has 2 solutions

$$z^2 = -1$$
$$\boxed{z = \pm i}$$

because $(i)^2 = i^2 = -1$
and $(-i)^2 = (-1)^2 \cdot i^2 = 1 \cdot -1 = -1$

b) Prove that the transformation f preserves the real axis. (i.e., if $R \subseteq \mathbb{C}$ denotes the real axis, prove that $f(R) = R$).

The set R consists of all complex numbers such that x is a real number:

$$R = \{x + 0i \in \mathbb{C} : x \in \mathbb{R}\}$$

We cannot simply say, $F(R) = 5R + \frac{4}{R}$.
This is not true!

We need to prove:

$$F(R) = \{ z \in \mathbb{C} : \exists x \in R \text{ s.t. } z = F(x) \}$$

* This is the range or Image (F).

Let $x \in R$, then $F(x) = 5x + \frac{4}{x}$

* Using an arbitrary real number $x + 0i$

By closure of the real numbers, $5x$ is real, $\frac{4}{x}$ is real. Therefore, the sum $5x + \frac{4}{x}$ is real.

Since $F(R)$ is real, F preserves the real axis.

$$F(R) \subseteq R$$

c) Prove that the transformation F does not preserve the line $y = x$.

$$\boxed{\mathbb{R}^2 \longleftrightarrow \mathbb{C} \quad (x, y) \quad x + yi}$$

$$L = \{ x + iy : x, y \in \mathbb{R}, x = y \}$$
$$= \{ x + ix : x \in \mathbb{R}, x \neq 0 \}$$

To proof something is not true,
we need to provide a counterexample.

If we choose $x = y = 1$ as an example,
Then $z = x + yi = x + xi = 1 + i$.

$$\boxed{z = 1 + i}$$

$$f(1+i) = 5(1+i) + \frac{4}{1+i} = \underline{5+5i} + \boxed{\frac{4}{1+i}}$$

Here we see that $5+5i$ <u>is</u> on the line $y=x$; however, we have something else $\left(\frac{4}{1+i}\right)$ being added to it.

<u>Check:</u>

$$5+5i + \frac{4}{1+i} \cdot \frac{1-i}{1-i}$$ — Multiplying by the conjugate in order to have a <u>real</u> denominator.

$$= 5+5i + \frac{4-4i}{(1+i)(1-i)}$$

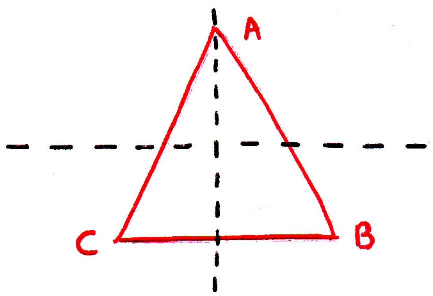
$$= 5+5i + \frac{4-4i}{1-i+i-i^2}$$ — $1 \cancel{-i} \cancel{+i} -(-1)$

$$= 5+5i + \frac{4-4i}{2}$$

$$= 5+5i + 2-2i$$

$$= 7+3i$$ which is not on the line $y=x$ because $3 \neq 7$.

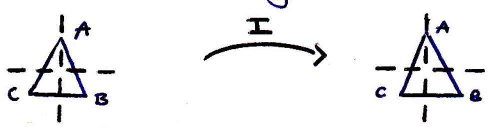
2. Consider an equilateral triangle T sitting in the complex plane $\mathbb{C}$, centered at the origin. Write down all the transformations <u>of the plane</u> which preserve the triangle. (<u>This means find all transformations of the plane f which satisfies F(T) = T</u>).



What transformations send T to itself?

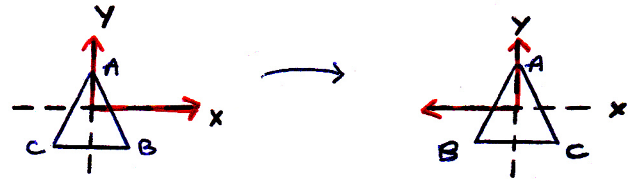
The transformations of the plane need to be isometric. This means they must be translations, rotations, and/or reflections. However, if we translate the plane, the triangle won't be centered at the origin anymore.

① <u>Identity</u>: The transformation that will evidently preserve the triangle is the identity transformation.



② <u>Reflection</u>: There are two different reflections that could be made to preserve the triangle.
(i) over the y-axis, (ii) over a bisector

(i) Reflection over y-axis

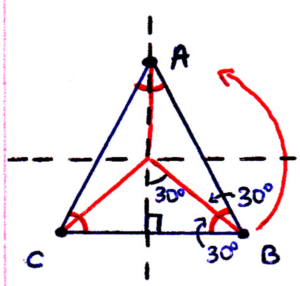


As we can see, even though the plane was reflected over the y-axis, the triangle remained intact.

(ii) Reflection over bisector (fixing point B)

(fixing point C)

③ Rotation: we can rotate the plane (about the origin) by multiples of 120°.



* Why 120°?

- Since the triangle is equilateral, we know each angle is equal to 60°
- Point B is located at -30° from the origin (or $-\pi/6$).

Rotating counterclockwise, point B would be going to point A which is located at i.

$$R(z) = e^{i\theta} z \Rightarrow R(e^{-\pi/6}) = i$$

In order to find the angle of rotation from B to A, we use the fact that $i = e^{i\pi/2}$.

Therefore, $\underline{e^{i\theta} \cdot e^{-\pi/6}}$ must equal $\underline{e^{i\pi/2}}$

$$e^{i\theta} \cdot e^{-\pi/6} = i$$

$$e^{i\theta} \cdot e^{-\pi/6} = e^{i\pi/2}$$

$$e^{i\theta} = \frac{e^{i\pi/2}}{e^{-\pi/6}}$$

$$e^{i\theta} = e^{i(\pi/2 + \pi/6)}$$

$$e^{i\theta} = e^{i(2\pi/3)}$$

Therefore, the angle of rotation is $\frac{2\pi}{3}$ or $120°$. We can also rotate by multiples of $120°$ like <u>$240°$</u> or $360°$. However, rotating by $360°$ would just be the identity transformation.